

## QUASICONFORMAL MAPPINGS AND ROYDEN ALGEBRAS IN SPACE<sup>(1)</sup>

BY  
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**Abstract.** On every open connected set  $G$  in Euclidean  $n$ -space  $R^n$  and for every index  $p > 1$ , we define the Royden  $p$ -algebra  $M_p(G)$ . We use results by F. W. Gehring and W. P. Ziemer to prove that two such sets  $G$  and  $G'$  are quasiconformally equivalent if and only if their Royden  $n$ -algebras are isomorphic as Banach algebras. Moreover, every such algebra isomorphism is given by composition with a quasiconformal homeomorphism between  $G$  and  $G'$ . This generalizes a theorem by M. Nakai concerning Riemann surfaces. In case  $p \neq n$ , the only homeomorphisms which induce an isomorphism of the  $p$ -algebras are the locally bi-Lipschitz mappings, and for  $1 < p < n$ , every such isomorphism arises this way. Under certain restrictions on the domains, these results extend to the Sobolev space  $H_p^1(G)$  and characterize those homeomorphisms which preserve the  $H_p^1$  classes.

**Introduction.** In 1960 Nakai proved [12] that two Riemann surfaces are quasiconformally equivalent if and only if their Royden algebras are isomorphic. In this paper we characterize a class of homeomorphisms and a family of Banach algebras which extend this result to higher dimensions.

On each finite subset  $G$  of Euclidean  $n$ -space  $R^n$  we define the Royden  $p$ -algebra  $M_p(G)$  for arbitrary  $p > 1$ .  $M_p(G)$  is a commutative semisimple Banach algebra with identity. We then form the Gelfand compactification of  $G$  with respect to the Royden  $p$ -algebra and call it the Royden  $p$ -compactification of  $G$  [15].

We also define  $Q_p$ -mappings of  $G$  onto  $G'$  and characterize them to be exactly those homeomorphisms which induce by composition an isomorphism between the respective Royden  $p$ -algebras. Using the Gelfand theory we prove that for  $1 < p \leq n$  every such isomorphism is obtained by composition with a  $Q_p$ -mapping.

If  $G$  has finite measure we may consider  $M_p(G)$  to be a subset of the Sobolev space  $H_p^1(G)$ . With certain restrictions on the domains, we also prove that the  $Q_p$ -mappings are exactly those homeomorphisms which leave the  $H_p^1$  classes invariant.

Recently F. W. Gehring has shown that all  $Q_p$ -mappings are quasiconformal mappings. In fact for every  $p \neq n$  they are precisely the class of all bi-Lipschitz

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Presented to the Society, January 23, 1969; received by the editors December 2, 1969 and, in revised form, March 9, 1970.

*AMS 1970 subject classifications.* Primary 30A60, 46J15; Secondary 46E35, 31B15.

*Key words and phrases.* Royden algebra, quasiconformal mappings, Sobolev space, Royden compactification, bi-Lipschitz mapping,  $p$ -capacity, conformal capacity.

<sup>(1)</sup> This paper is based on the author's dissertation at Indiana University, May, 1969. The research was partially supported by NSF Grant Nos. GP-4071 and GP-7505.

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mappings. Thus the case  $p=n$  is distinguished when studying either the Royden algebras or the Sobolev spaces under composition. This special case was introduced by C. Loewner [8] as "conformal capacity" and studied by F. W. Gehring [4].

H. M. Reimann [14] has recently considered a class of homeomorphisms similar to the  $Q_p$ -mappings.

The author wishes to express appreciation to Professor George Springer for his inspiration as thesis advisor and to Professors Frederick Gehring and William Ziemer for their many helpful suggestions.

**1. Preliminaries.** The following notation is used:  $m_n$  is  $n$ -dimensional Lebesgue measure,  $m_{n-1}$  is  $(n-1)$ -dimensional Hausdorff measure,  $L_p(G)$  is the usual space of equivalence classes of functions  $f$  for which  $|f|^p$  is  $m_n$ -integrable, with norm  $\|f\|_p$ , and  $C_0^\infty(G)$  is the space of infinitely differentiable functions with compact support contained in  $G$ . Unless otherwise indicated, all functions are complex-valued and all integrals are taken over  $G$  (or  $G'$ ).  $x=(x^1, \dots, x^n)$  denotes an arbitrary point in  $R^n$ ,  $B(x, r)$  and  $S(x, r)$  are the usual ball and sphere, abbreviated  $B^n$  and  $S^{n-1}$  in case  $x=0$  and  $r=1$ , and  $\omega_n=m_{n-1}(S^{n-1})$ .

Denote by  $\text{ACL}(G)$  the collection of all functions which are absolutely continuous along the intersection of every compact  $n$ -interval with  $m_{n-1}$ -almost every line parallel to the coordinate axes. Note that every  $f \in \text{ACL}(G)$  has a gradient.

More generally, a function  $f$  has a (weak) gradient

$$\nabla f = (r^1, \dots, r^n)$$

if there exist distributional derivatives  $r^i$  satisfying

$$(1.1) \quad \int w r^i dm_n = - \int f \frac{\partial w}{\partial x^i} dm_n, \quad w \in C_0^\infty(G),$$

for  $i=1, \dots, n$ . Then the *Dirichlet  $p$ -integral* of  $f$  is

$$D_p[f] = \int |\nabla f|^p dm_n = \int \left[ \sum_{i=1}^n |r^i|^2 \right]^{p/2} dm_n.$$

In case  $f$  has locally integrable partial derivatives, they are equal  $m_n$ -a.e. to the  $r^i$ , and hence satisfy (1.1). Thus, for  $f \in \text{ACL}(G)$  with  $D_p[f] < \infty$ ,  $\nabla f$  as defined above is the usual gradient.

A real-valued function  $u$  is of class  $H_p^1$  on  $G$  if  $u \in L_p(G)$  and if  $u$  has a gradient and satisfies  $D_p[u] < \infty$ . A homeomorphism is of class  $H_p^1$  on  $G$  if each of its coordinate functions is.  $H_p^1(G)$  is the space of all such functions with the norm

$$\|u\| = \left\{ \int [|u|^2 + |\nabla u|^2]^{p/2} dm_n \right\}^{1/p}.$$

As is customary with  $L_p(G)$  we write  $u \in H_p^1(G)$  and mean that  $u$  is of class  $H_p^1$  on  $G$ .  $H_p^1(G)$  is complete in this norm, and each of its equivalence classes contains a

representative in  $\text{ACL}(G)$  [10, p. 66]. This fact together with Fubini's theorem implies the following.

**LEMMA 1.1.** *A continuous real-valued function  $u$  which is locally of class  $H_p^1$  on  $G$  is in  $\text{ACL}(G)$ .*

A ring  $R$  is a connected open set in  $R^n$  whose complement consists of a bounded component  $C_0$  and an unbounded one  $C_1$ . For  $1 < p < \infty$ , the  $p$ -capacity of  $R$  is

$$\text{cap}_p R = \inf \left\{ \int_R |\nabla u|^p dm_n \right\},$$

where the infimum is taken over all continuous functions  $u \in \text{ACL}(R)$  with constant boundary values 0 and 1. As in [3, p. 358] each such function may be extended to  $R^n$  and the integral taken over  $R^n$  without affecting  $\text{cap}_p R$ . Such an extended function is *admissible* for  $R$ .

In case  $R$  is the spherical ring  $\{x : a < |x - x_0| < b\}$ , then

$$(1.2) \quad \begin{aligned} \text{cap}_p R &= \omega_n \left\{ \int_a^b t^{q-1} dt \right\}^{1-p} = \omega_n (\log b/a)^{1-n}, & p = n, \\ &= \omega_n ((b^q - a^q)/q)^{1-p}, & p \neq n, \end{aligned}$$

where  $q = (p-n)/(p-1)$ . Note that this gives easily the existence of a unique extremal function for  $\text{cap}_p R$  whenever  $R$  is a spherical ring; for  $p=n$  this is well known and for  $p \neq n$  it is the function  $|x|^q$  with proper normalization. The uniqueness follows from Clarkson's inequality as in [11, pp. 74, 77].

**2. The Royden  $p$ -algebra.** Denote by  $M_p(G)$  the collection of all continuous functions  $f$  such that

- (i)  $f \in \text{ACL}(G)$ ,
- (ii)  $\|f\|_\infty = \sup_G |f| < \infty$ ,
- (iii)  $D_p[f] < \infty$ .

Since Minkowski's inequality implies

$$(2.1) \quad D_p[fg]^{1/p} \leq \|f\|_\infty D_p[g]^{1/p} + \|g\|_\infty D_p[f]^{1/p},$$

it follows that  $M_p(G)$  is an algebra under pointwise multiplication with the constant function 1 as identity.

**THEOREM 2.1.** *With the norm*

$$(2.2) \quad \|f\| = \|f\|_\infty + D_p[f]^{1/p},$$

$M_p(G)$  is a complex commutative Banach algebra with identity.

**Proof.** We conclude from (2.1) that  $\|fg\| \leq \|f\| \|g\|$ , and since  $\|1\| = 1$  we only need to show that  $M_p(G)$  is complete in the norm (2.2) to finish the proof.

Let  $\{f_j\}$  be a Cauchy sequence in  $M_p(G)$ . Then  $\{f_j\}$  converges uniformly on  $G$  to a bounded continuous function  $f$ , and the derivatives  $\{f_j'\}$  form a Cauchy sequence in

$L_p(G)$ ,  $i = 1, \dots, n$ . There exist functions  $r^1, \dots, r^n \in L_p(G)$  such that  $\|r_j^i - r^i\|_p \rightarrow 0$ , and an easy calculation shows that  $f$  and  $r^i$  satisfy (1.1). Thus  $f$  has a gradient satisfying  $D_p[f] < \infty$ , and  $f$  is locally of class  $H_p^1$  on  $G$ . Finally Lemma 1.1 implies that  $f \in M_p(G)$ , proving the theorem.

A sequence  $\{f_j\}$  of functions in  $M_p(G)$  converges in the  $BD_p$ -topology to  $f$  if

- (i)  $\|f_j - f\|_\infty \leq M < \infty$ ,
- (ii)  $f_j \rightarrow f$  uniformly on compact subsets of  $G$ ,
- (iii)  $D_p[f_j - f] \rightarrow 0$ .

**COROLLARY 2.2.**  $M_p(G)$  is complete in the  $BD_p$ -topology.

**Proof.** Any Cauchy sequence in the  $BD_p$ -topology converges uniformly on compact subsets to a bounded continuous function, and the proof is similar to the above proof.

**3. Induced isomorphisms.** Let  $T$  be a homeomorphism of  $G$  onto  $G'$ . Denote  $T \in Q_p(K) = Q_p(K; G)$  if for some  $K < \infty$

$$\text{cap}_p T(R) \leq K \text{cap}_p R$$

for every spherical ring  $R$  satisfying  $\bar{R} \subset G$ .  $T$  is called a  $Q_p$ -mapping if  $T \in Q_p(K; G)$  and  $T^{-1} \in Q_p(K; G')$  for some  $K$ .  $T$  is a *quasiconformal mapping* if for some  $1 \leq K < \infty$

$$K^{-1} \text{cap}_n R \leq \text{cap}_n T(R) \leq K \text{cap}_n R$$

for every ring  $R$  satisfying  $\bar{R} \subset G$ .

The restriction to spherical rings in the definition of  $Q_p(K)$  is necessary for the additivity of the  $p$ -capacity in (4.3).

**REMARK 3.1.** It is well known that a  $Q_n$ -mapping is a quasiconformal mapping (see [3], [11] or [19]). Thus every  $Q_p$ -mapping is a quasiconformal mapping [5, Theorem 1] and is in fact a bi-Lipschitz mapping for  $p \neq n$  [5, Theorem 2]. It follows easily (see Corollary 3.3) that the class of  $Q_p$ -mappings for all  $p \neq n$  is precisely the class of all homeomorphisms  $T$  which together with their inverses satisfy the following Lipschitz condition: There exists an  $M < \infty$  such that

$$(3.1) \quad \limsup_{|x-y| \rightarrow 0} \frac{|T(x) - T(y)|}{|x - y|} \leq M, \quad x \in G.$$

**THEOREM 3.2.** Every  $Q_p$ -mapping  $T$  of  $G$  onto  $G'$  induces an algebra isomorphism  $\varphi_T$  of  $M_p(G')$  onto  $M_p(G)$  defined by

$$(3.2) \quad \varphi_T(f) = f \circ T, \quad f \in M_p(G').$$

**Proof.** Since the composition  $\varphi_T$  clearly preserves the algebraic operations, we prove only  $\varphi_T(M_p(G')) \subset M_p(G)$ . This will complete the proof since the opposite inclusion follows similarly.

Let  $f \in M_p(G')$  and set  $u = \operatorname{Re} f$  and  $v = u \circ T$ . In case  $p \neq n$  it follows from (3.1) and the Rademacher-Stepanoff theorem [17, p. 310] that  $T$  and  $T^{-1}$  are differentiable  $m_n$ -a.e. and that

$$(3.3) \quad M^{-n} \leq |JT| \leq M^n \quad m_n\text{-a.e. in } G,$$

for some  $M < \infty$ . Again using (3.1), we get

$$(3.4) \quad |\nabla v| \leq M |\nabla u| \circ T \quad m_n\text{-a.e. in } G.$$

Then (3.3) and (3.4) yield

$$\begin{aligned} \int_G |\nabla v|^p dm_n &\leq M^p \int_G |\nabla u|^p \circ T dm_n \\ &\leq M^{p+n} \int_G |\nabla u|^p \circ T |JT| dm_n = M^{p+n} \int_G |\nabla u|^p dm_n. \end{aligned}$$

For  $p = n$  let  $Q$  be any  $n$ -cube satisfying  $\bar{Q} \subset G$  and denote  $Q' = T(Q)$ ,  $u_0 = u|_{Q'}$  and  $v_0 = u_0 \circ T|_Q$ . Then  $u_0 \in H_n^1(Q')$  and it follows from [20, Remark 4.2] that  $v_0 \in H_1^1(Q)$  and

$$\int_Q |\nabla v_0|^n dm_n \leq K \int_{Q'} |\nabla u_0|^n dm_n.$$

Exhausting  $G$  by a sequence of disjoint cubes gives

$$\int_G |\nabla v|^n dm_n \leq K \int_{G'} |\nabla u|^n dm_n.$$

We apply the above argument to  $\operatorname{Im} f$  and see that for any  $1 < p < \infty$  there exists a constant  $K_2$  depending only on  $K, M, p$  and  $n$  such that

$$(3.5) \quad D_p[g] \leq K_2 D_p[f],$$

where  $g = f \circ T$ ,  $f \in M_p(G')$ . Lemma 1.1 then implies that  $g \in M_p(G)$ , concluding the proof.

**COROLLARY 3.3.** *Every homeomorphism  $T$  of  $G$  onto  $G'$  such that  $T$  and  $T^{-1}$  satisfy (3.1) is a  $Q_p$ -mapping for all  $p$ .*

**Proof.** Such a  $T$  satisfies (3.3) and (3.4). Let  $R'$  be a spherical ring in  $G'$ ,  $\bar{R} \subset G'$ , and let  $u$  be the extremal function for  $\operatorname{cap}_p R'$ . Setting  $v = u \circ T$ , the above proof gives

$$\operatorname{cap}_p R \leq \int_G |\nabla v|^p dm_n \leq M^{p+n} \int_{G'} |\nabla u|^p dm_n = M^{p+n} \operatorname{cap}_p R',$$

where  $R = T^{-1}(R')$ . Hence  $T^{-1} \in Q_p(M^{p+n})$  and the corollary follows by applying the same argument to  $T$ .

**4. Characterization of  $Q_p$ -mappings.** In this section we give a necessary and sufficient condition for a given homeomorphism  $T$  of  $G$  onto  $G'$  to be a  $Q_p$ -mapping.

Suppose that  $T$  has the property that  $u = v \circ T^{-1}$  is admissible for  $T(R)$  whenever  $v$  is the extremal function for  $\text{cap}_p R$  for any spherical ring satisfying  $\bar{R} \subset G$ ; i.e., suppose  $T^{-1}$  preserves ACL functions. Suppose further that there exists a constant  $K_1 < \infty$  independent of  $R$  such that

$$(4.1) \quad D_p[u]^{1/p} \leq K_1 \|v\|$$

whenever  $u$  and  $v$  are as above.

LEMMA 4.1.  $T \in Q_p(K)$  for some  $K \leq K_1^p$ .

**Proof.** If the lemma were false there would exist a spherical ring  $R = \{x : a < |x - x_0| < b\}$  with  $\bar{R} \subset G$  such that

$$(4.2) \quad \text{cap}_p T(R) \geq d^p \text{cap}_p R$$

for some  $d > K_1$ . We may assume that  $\text{cap}_p R$  is arbitrarily large because for any positive integer  $m$  we may define

$$R_j = \{x : r_{j-1} < |x - x_0| < r_j\}, \quad j = 1, \dots, m,$$

where

$$\begin{aligned} r_j &= [a^{m-j} b^j]^{1/m}, & p &= n, \\ &= [((m-j)a^q + jb^q)/m]^{1/q}, & p &\neq n, \end{aligned}$$

and  $q = (p-n)/(p-1)$ . Then  $\text{cap}_p R_j = m^{p-1} \text{cap}_p R$  and

$$(4.3) \quad \sum_{j=1}^m (\text{cap}_p R_j)^{1/(1-p)} = (\text{cap}_p R)^{1/(1-p)},$$

which together with the  $p$ -capacity version of [2, Lemma 2] gives

$$d^p \text{cap}_p R \leq \left\{ \sum_{j=1}^m (\text{cap}_p T(R_j))^{1/(1-p)} \right\}^{1-p}$$

An easy calculation shows that for some  $1 \leq k \leq m$  we must have

$$\text{cap}_p T(R_k) \geq m^{p-1} d^p \text{cap}_p R = d^p \text{cap}_p R_k,$$

and  $R_k$  satisfies both (4.2) and  $\text{cap}_p R_k = m^{p-1} \text{cap}_p R$ . Thus we could use  $R_k$  instead of  $R$ .

Now choose some  $c$ , with  $K_1 < c < d$ , and assume

$$\text{cap}_p R > c^p (d-c)^{-p}.$$

Let  $v$  be the extremal function for  $\text{cap}_p R$  and set  $u = v \circ T^{-1}$ . Then by hypothesis  $D_p[u] \geq \text{cap}_p T(R)$ , and it follows from (4.1) and (4.2) that

$$\begin{aligned} K_1 \|v\| &\geq D_p[u]^{1/p} \geq (\text{cap}_p T(R))^{1/p} \geq d(\text{cap}_p R)^{1/p} \\ &= (d-c)(\text{cap}_p R)^{1/p} + c D_p[v]^{1/p} > c(1 + D_p[v]^{1/p}) = c \|v\|. \end{aligned}$$

But this gives  $c < K_1$ , contradicting the choice of  $c > K_1$  and proving the lemma.

**THEOREM 4.2.** *Every homeomorphism  $T$  of  $G$  onto  $G'$  for which  $\varphi_T$  is an algebra isomorphism of  $M_p(G')$  onto  $M_p(G)$  is a  $Q_p$ -mapping.*

**Proof.** Let  $R$  be a spherical ring,  $\bar{R} \subset G$ , and let  $v$  be the extremal function for  $\text{cap}_p R$ . Then  $\varphi_T^{-1}(v) = v \circ T^{-1} = u \in M_p(G')$ , and  $u$  is clearly admissible for  $T(R)$ . In §6 we apply some Banach algebra results (Lemma 6.1) to prove that  $\varphi_T^{-1}$  is a bounded operator. Setting  $K_1 = \|\varphi_T^{-1}\|$  satisfies (4.1) and by Lemma 4.1 we have  $T \in Q_p(K)$  for some  $K \leq K_1^p$ . Applying the same argument to  $T$  completes the proof.

Theorems 3.2 and 4.2 give the following characterization of a  $Q_p$ -mapping.

**COROLLARY 4.3.** *Let  $T$  be a homeomorphism of  $G$  onto  $G'$ . Then  $T$  is a  $Q_p$ -mapping if and only if  $\varphi_T$  is an algebra isomorphism of  $M_p(G')$  onto  $M_p(G)$ .*

Another corollary follows from Theorem 4.2 and Remark 3.1.

**COROLLARY 4.4.** *Let  $T$  be a  $Q_p$ -mapping of  $G$  onto  $G'$ ,  $p \neq n$ . Then  $M_p(G)$  and  $M_{p'}(G')$  are algebra isomorphic for every  $p'$ .*

**COROLLARY 4.5.**  *$\varphi_T$  is an isometry if and only if  $T, T^{-1} \in Q_p(1)$ .*

**Proof.** If  $T, T^{-1} \in Q_p(1)$  then it follows that the constant  $K_2 = 1$  in (3.5). For  $p = n$  this is well known and for  $p \neq n$  it follows from Remark 3.1 and the fact that  $K_0 = K^{2n/p}$  in the proof of [5, Theorem 1]. Then (3.5) applied to  $T$  and  $T^{-1}$  implies that  $D_p[f] = D_p[\varphi_T(f)]$ ,  $f \in M_p(G')$ . But  $\|f\|_\infty = \|\varphi_T(f)\|_\infty$  trivially, proving the isometry.

Conversely, if  $\varphi_T$  is an isometry then (4.1) holds with  $K_1 = 1$  and Lemma 4.1 implies  $T \in Q_p(1)$ . The same is true for  $T^{-1}$ , concluding the proof.

**5. Sobolev spaces.** With certain geometrical restrictions on the domains  $G$  and  $G'$  the techniques of the previous section may be used to characterize those homeomorphisms whose composition maps the Sobolev spaces  $H_p^1(G)$  and  $H_p^1(G')$  onto each other.

Let  $G_1$  be a convex subdomain of  $G$ .  $G$  is *star-shaped with respect to  $G_1$*  if  $G$  contains every cone whose vertex is in  $G_1$  and whose generators terminate on  $G_1$ , i.e., if  $G$  is star-shaped with respect to every point of  $G_1$ . We first give a Sobolev imbedding lemma.

**LEMMA 5.1.** *Let  $G$  be a domain which is star-shaped with respect to some convex subdomain and which satisfies  $m_n(G) < \infty$ . Then there exists a constant  $M$ , depending only on  $p$  and  $G$ , such that every  $v \in H_p^1(G)$  satisfies*

$$(5.1) \quad \|v\|_p \leq M(D_p[v]^{1/p} + m_n(G)^{-1} \|v\|_1).$$

**Proof.** The cases  $1 < p \leq n$  and  $p > n$  are just special cases of Theorems 2 and 1, respectively, of [18, pp. 56, 57]. Cf. also [7, pp. 369–380].

**REMARK 5.2.** Actually (5.1) holds for more general domains than those considered in Lemma 5.1. For example [7, Remark 4 and Theorem 2, p. 376], if

$G = G_1 \cup G_2$ , where (5.1) holds for  $G_1$  and  $G_2$ , and if  $m_n(G_1 \cap G_2) > 0$ , then (5.1) holds for  $G$ . See [10, Theorem 3.2.1 and pp. 72–74] for a still more general class of domains for which (5.1) holds.

**THEOREM 5.3.** *Let  $G$  and  $G'$  be domains of finite measure for which (5.1) holds, and let  $T$  be a homeomorphism of  $G$  onto  $G'$ . Then  $T$  is a  $Q_p$ -mapping if and only if  $\varphi_T$  maps  $H_p^1(G')$  onto  $H_p^1(G)$ .*

**Proof.** If  $T$  is a  $Q_p$ -mapping we use Ziemer's theorem [20] to prove that composition with  $T$  preserves the  $H_p^1$  classes. In case  $p \neq n$  we conclude from [5, Lemma 7] and Remark 3.1 that  $T$  is bi-measurable. (3.1) implies that  $\|dT^{-1}\| \leq M$ , hence that  $T^{-1}$  is of class  $H_{p'}^1$  on  $G$ , where  $p' = p(n-1)/(p-1)$ . Then for any  $1 < p < \infty$  it follows from [20, Theorem 1.1], as in the proof of (3.5) for  $p = n$ , that  $D_p[v] \leq K_2 D_{p'}[u] < \infty$  and  $v \in H_p^1(G)$  whenever  $v = u \circ T$ ,  $u \in H_{p'}^1(G')$ . Thus  $\|v\|_1 < \infty$  and  $D_p[v] < \infty$ , and (5.1) implies  $\|v\|_p < \infty$ . Finally this means  $v \in H_p^1(G)$  and  $\varphi_T(H_{p'}^1(G')) \subset H_p^1(G)$ . The opposite inclusion follows similarly, proving half of the theorem.

Conversely, if  $\varphi_T$  maps  $H_p^1(G')$  onto  $H_p^1(G)$ , then by Lemma 1.1 its restriction is actually an isomorphism between the Royden  $p$ -algebras, and the other half follows from Theorem 4.2.

**REMARK 5.4.** The boundedness of the partial derivatives of both  $T$  and  $T^{-1}$  is known to be sufficient for  $T$  to preserve the  $H_p^1$  classes [10, Theorems 3.1.5 and 3.1.6]. Theorem 5.3 shows that this condition is also necessary for  $p \neq n$ . It is not necessary for  $p = n$ , since the partial derivatives of a quasiconformal mapping need not be bounded.

**6. The Royden  $p$ -compactification.** We recall first some well-known facts about Banach algebras. Let  $A$  be a normed algebra of continuous functions defined on  $G$  which contains the constant functions. Suppose that  $A$  is *regular*, i.e., that for every closed set  $W \subset G$  and every  $x \in G - W$  there exists some  $f \in A$  such that  $f = 0$  on  $W$  and  $f(x) \neq 0$ .

Denote by  $G^* = G_A^*$  the collection of all nonzero bounded complex linear functionals  $\chi$  on  $A$  which satisfy

$$\chi(fg) = \chi(f)\chi(g) \quad \text{and} \quad \chi(\bar{f}) = \overline{\chi(f)}, \quad f, g \in A.$$

Then  $\|\chi\| = 1$  for every  $\chi \in G^*$ , and  $G^*$  is contained in the unit sphere of the dual space of  $A$ , inheriting the relative weak\* topology generated by  $A$ . That is,  $\chi_\alpha \rightarrow \chi$  in  $G^*$  if and only if

$$\lim_{\alpha} |\chi_\alpha(f) - \chi(f)| = 0, \quad f \in A.$$

In this topology  $G^*$  is closed, and hence is a compact Hausdorff space by Alaoglu's theorem [16, p. 202]. For each  $x \in G$  define

$$\hat{x}(f) = f(x), \quad f \in A.$$



Then since  $A$  is regular,  $x \rightarrow \hat{x}$  is a homeomorphism of  $G$  onto a subset  $\hat{G} = \hat{G}_A$  of  $G^*$ .

Suppose in addition that  $A$  is *selfadjoint* and *inverse-closed*, i.e., that  $f \in A$  implies  $\bar{f} \in A$  and that  $f \in A$  and  $\inf_G |f| > 0$  imply  $1/f = f^{-1} \in A$ , respectively. For each  $f \in A$  define

$$\hat{f}(\chi) = \chi(f), \quad \chi \in G^*.$$

Then  $f \rightarrow \hat{f}$  is a homomorphism of  $A$  onto a subset  $\hat{A}$  of  $C(G^*)$ , which by the Stone-Weierstrass theorem is dense. It follows [12, p. 163] that  $\hat{G}$  is dense in  $G^*$  and that

$$(6.1) \quad G^* - \hat{G} = \{\chi \in G^* : \chi(f) = 0, f \in A_0\},$$

where  $A_0$  denotes those functions in  $A$  with compact support in  $G$ . Since  $G^* - \hat{G}$  is thus the intersection of a family of zero sets of continuous functions,  $\hat{G}$  is open in  $G^*$  and  $\Delta = \Delta_A = G^* - \hat{G}$  is called the *A-ideal boundary* of  $G$ . For every  $x \in G$ ,  $\hat{f}(\hat{x}) = \hat{x}(f) = f(x)$ , and by identifying  $G$  with its homeomorphic image  $\hat{G}$  we may consider  $\hat{f}$  to be a continuous extension of  $f$  to  $G^*$ . Then  $G^* = G_A^*$  is the *A-compactification* of  $G$  [1, Chapter 9] (which is unique up to a homeomorphism which leaves  $G$  fixed).

Note that  $M_p(G)$  is regular since it contains  $C_0^\infty(G)$ . That  $M_p(G)$  is selfadjoint is trivial, and it is easy to verify that it is inverse-closed. Thus we may apply the above theory to the Royden  $p$ -algebra, in which case  $\Delta$  and  $G^*$  are called the *Royden  $p$ -ideal boundary* and the *Royden  $p$ -compactification*, respectively, of  $G$ .

Since  $M_p(G)$  separates points, it is semisimple, i.e.,  $f \rightarrow \hat{f}$  is one-to-one. Thus we have the following result [9, p. 76] which has already been used in the proof of Theorem 4.2.

**LEMMA 6.1.** *Let  $\psi$  be an algebra homomorphism from a commutative Banach algebra onto  $M_p(G)$ . Then  $\psi$  is a bounded linear operator.*

**LEMMA 6.2.** *For  $1 < p \leq n$ , no point of the Royden  $p$ -ideal boundary  $\Delta$  has a countable neighborhood basis.*

**Proof.** Assuming the contrary as in [13, p. 558], let  $\{U_j\}$  be a countable neighborhood basis for the topology at  $\chi \in \Delta$  and let  $\hat{V}_j = U_j \cap \hat{G}$ . Then  $\{V_j\}$  is a sequence of nonempty open subsets of  $G$ , and we may assume that  $\bar{V}_{j+1} \subset V_j$ ,  $j = 1, 2, \dots$ . For each  $j$ ,  $V_j - \bar{V}_{j+1}$  contains some ball  $B(x_j, b_j)$ . Define  $R_j = B(x_j, b_j) - \text{Cl}(B(x_j, a_j))$ , where

$$\begin{aligned} a_j &= b_j \exp -(2^j \omega_n)^{1/(n-1)}, & p &= n, \\ &= b_j [1 - qb_j^{-q} (2^j \omega_n)^{1/(p-1)}]^{1/q}, & p &< n; \end{aligned}$$

then  $\text{cap}_p R_j = 2^{-j}$ ,  $j = 1, 2, \dots$

Denote the extremal function for  $\text{cap}_p R_j$  by  $v_j$  and set

$$w_k(x) = \sum_{j=1}^k v_j(x) \quad \text{and} \quad w(x) = \sum_{j=1}^{\infty} v_j(x).$$

Since we may choose  $v_j=0$  on  $G-B(x_j, b_j)$  and since  $\{B(x_j, b_j)\}$  are all disjoint,  $\{w_k\}$  converges uniformly on compact subsets of  $G$  to  $w$ , and  $0 \leq w \leq 1$ . Also  $D_p[w_k - w] = 2^{-k} \rightarrow 0$ , and  $w_k \rightarrow w$  in the  $BD_p$ -topology. Then  $w \in M_p(G)$  by Corollary 3.2, and  $w$  has a continuous extension to  $G^*$ . Choosing  $y_j \in S(x_j, b_j)$  gives  $\hat{y}_j \rightarrow \chi$  with  $\hat{w}(\hat{y}_j) = 0$ . But  $\hat{x}_j \rightarrow \chi$  also and  $\hat{w}(\hat{x}_j) = 1$ , contradicting the continuity of  $\hat{w}$  and proving the lemma.

**7. Induced homeomorphisms.** Let  $\psi$  be an algebra isomorphism of  $M_p(G')$  onto  $M_p(G)$ . Then the adjoint mapping  $T^*$ , defined by

$$T^*(\chi) = \chi \circ \psi, \quad \chi \in G^*,$$

is a homeomorphism of  $G^*$  onto a closed subset of  $G'^*$  [9, p. 76]. Since  $\psi^{-1}$  satisfies the same conditions it is clear that  $T^*(G^*) = G'^*$ .

Suppose that  $T^*(\hat{G}) = \hat{G}'$ . Then composition with  $x \rightarrow \hat{x}$  and its inverse would induce a homeomorphism  $T$  of  $G$  onto  $G'$  satisfying

$$(7.1) \quad T(x) = y, \quad \text{where } \hat{y} = T^*(\hat{x}).$$

Note that the isomorphism  $\psi$  would be given by composition with  $T$ , for let  $f \in M_p(G')$ . Then for any  $x \in G$

$$(7.2) \quad \psi(f)(x) = \hat{x} \circ \psi(f) = T^*(\hat{x})(f) = \hat{y}(f) = f \circ T(x),$$

where  $y = T(x)$ .

**THEOREM 7.1.** *For  $1 < p \leq n$ , every isomorphism  $\psi$  of  $M_p(G')$  onto  $M_p(G)$  induces a  $Q_p$ -mapping  $T$  of  $G$  onto  $G'$  such that  $\psi = \varphi_T$ .*

**Proof.** For  $1 < p \leq n$ , Lemma 6.2 implies that  $T^*(\hat{G}) \subset \hat{G}'$  and that  $T^{*-1}(\hat{G}') \subset \hat{G}$ . Thus the above argument holds, and (7.2) gives  $\psi = \varphi_T$ . Theorem 4.2 concludes the proof.

**COROLLARY 7.2.** *For  $1 < p \leq n$  there is a one-to-one correspondence between  $Q_p$ -mappings of  $G$  onto  $G'$  and algebra isomorphisms of  $M_p(G)$  onto  $M_p(G')$  given by  $T \rightarrow \varphi_T^{-1}$ .*

**COROLLARY 7.3.**  *$G$  and  $G'$  are quasiconformally equivalent if and only if  $M_p(G)$  and  $M_p(G')$  are algebra isomorphic for some  $p$  satisfying  $1 < p \leq n$ .*

**REMARK 7.4.** Using [6, §10] we may now observe that  $M_3(G)$  and  $M_3(B^3)$  are not isomorphic if the boundary of  $G$  has an inward directed spire or an outward directed ridge.

**8. The Royden  $p$ -ideal boundary.** We conclude with some remarks about the behavior of  $\Delta$  under  $Q_p$ -mappings defined on  $G$ . The Royden  $p$ -harmonic boundary  $\Gamma$  is defined by

$$\Gamma = \{\chi \in G^* : \chi(f) = 0, f \in M_\Delta(G)\},$$

where  $M_\Delta(G)$  denotes the closure in the  $BD_p$ -topology of the functions in  $M_p(G)$  with compact support in  $G$ . It follows from Corollary 2.2 and (6.1) that  $\Gamma$  is a closed subset of  $\Delta$ .

**THEOREM 8.1.** *Every  $Q_p$ -mapping  $T$  of  $G$  onto  $G'$  induces a homeomorphism of  $\Delta$  onto  $\Delta'$  whose restriction to  $\Gamma$  is a homeomorphism onto  $\Gamma'$ .*

**Proof.** By Theorem 3.2,  $T$  induces an isomorphism  $\varphi_T$  of  $M_p(G')$  onto  $M_p(G)$  which in turn induces a homeomorphism  $T^*$  of  $G^*$  onto  $G'^*$  defined by

$$T^*(\chi) = \chi \circ \varphi_T, \quad \chi \in G^*.$$

But for  $x \in G$

$$\begin{aligned} T^*(\hat{x})(f) &= \hat{x} \circ \varphi_T(f) \\ &= \hat{x}(f \circ T) = f(y) = \hat{y}(f) \end{aligned}$$

for all  $f \in M_p(G')$ , where  $y = T(x)$ . Thus  $T^*(\hat{G}) \subset \hat{G}'$ , and the opposite inclusion implies that the restriction of  $T^*$  to  $\Delta$  is the desired homeomorphism.

Now let  $\chi \in \Gamma$  and let  $\{f_n\}$  be a sequence of functions in  $M_p(G)$  with compact support which converges in the  $BD_p$ -topology to  $f$ . We show that  $T^*(\chi)(f) = 0$ . Denote  $g_n = f_n \circ T$  and  $g = f \circ T$ . Then (3.5) implies that  $D_p[g_n - g] \rightarrow 0$ , and  $\{g_n\}$  converges to  $g$  in the  $BD_p$ -topology. But each  $g_n$  has compact support and

$$T^*(\chi)(f) = \chi \circ \varphi_T(f) = \chi(g) = 0.$$

Thus  $T^*(\Gamma) \subset \Gamma'$ , and the opposite inclusion completes the proof.

**Added in proof.** H. M. Reimann has recently made me aware of a simple example which shows that Theorem 7.1 is false for  $p > n$ .

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